Liquidity Trap and Persistent Non-Walrasian Equilibria in a Fixed Price Economy

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Abstract: *This paper treats a fixed price economy and purports to illustrate the IS-LM schedules from a general equilibrium point of view. Both pegging a rate of interest and direct controling money stock are considered as monetary policies and their invalidity under a liquidity trap is shown in a fixed price economy in addition to persistency of non-Walrasian equilibria.*

I. INTRODUCTION

Since Keynes [8] casually exhibited a liquidity trap in the *General Theory*, the invalidity of monetary policy under the trap has been established in fact as one of main propositions in macroeconomics. Following it, from a microeconomic point of view Patinkin [10], subsequently Grandmont and Laroque [4] and Hool [7] also showed the liquidity trap as an infinite demand for money which appeared as a result of the limit of a sequence of equilibria. As a matter of fact, however, there are some difficulties in such microeconomic illustrations of a liquidity trap, according as Grandmont [3] himself pointed out some problems in temporary equilibrium models.

Firstly the problem of production must be considered. As long as the world of the General Theory is indispensably painted by production, it is inappropriate to suppose no production and consequently to make an ambiguous point of the relation between production and speculation. Secondly, connected with the first problem, Grandmont and Laroque [4] imposed an *ad hoc* condition on financing (or bond issuing) so that only a monetary authority (Bank) could issue new bonds. To this Hool [7] exhibited a model allowing agents to issue bonds for finance by themselves. He supposed, however, that there was no debt carried over from the preceding dates. In any economy where bonds are issued or portfolios are selected under speculation to the future, the problem of existing debts may be inevitable to consider. Thirdly, what is most important, the invalidity of monetary policy under a liquidity trap is not as yet explained directly. Only the proof of an infinite demand for money is insufficient to show the invalidity and, on the contrary, there may be no limitation in economic factors that prevents a monetary authority from mainpulating in such a case a rate of interest to fall toward zero and therefore from realizing full employment.

This paper will develop by considering these three points, in particular concentrating on the third one. In this there is supposed an economy with production and a monetary authority (Bank), in which financing and speculative actions of agents are considered. Each agent is able to borrow money limitlessly (*i.e.* issue unbounded bonds) at a prevailing rate of interest, provided that he will not almost default in the future. Bank has two alternative monetary policies: the one is to predetermine a rate of interest as a target, according to which money supply is determined so as to equilibrate money demand and supply, while the other is to control directly money supply by putting the level of money stock as a target. Also there may be debts carried over from the past, which are not great enough to cause inevitable defaults. Such an assumption is required for simplicity that any agent's issuing bonds are accepted as homogenous ones[1].

A further thing to remark is to suppose an economy with fixed price markets according to the so-called IS-LM schedules. Prices are predetermined at each temporary date, during which actual demands and supplies are balanced by quantitative adjustments. Considering this fixed price economy as a microeconomic illustration of the IS-LM schedules, the primary subject is to expose the existence of fixed-price equilibria, a liquidity trap and the invalidity of monetary policy under the trap.

In the following three sections the actions of consumers, producers and Bank are introduced respectively. In Section 5 we show the existence of an unemployment (non-Walrasian) equilibrium and subsequently a liquidity trap and persistency of non-Walrasian equilibria. In Section 6 concluding remarks are provided and last the proofs of main propositions are gathered.

II. CONSUMERS

Let I be the set of consumers. In the economy where consumers act there exist l kinds of goods, bond and money. Let $e_i=(x_i^0, b_i^0, m_i^0)$ be consumer i's endowment which is composed of his initially holding goods, bond and money, respectively; we assume $(x_i^0, m_i^0)\in R_+^{l+1}$, $b_i^0\in R$ and $\sum_{i\in I}m_i^0>0$. Let (x_i^1, b_i, m_i) be a bundle of consumer i's desiring goods, bond and money, respectively.

In goods markets prices are predetermined at the beginning of a date and quantity constraints on purchases and sales, denoted by $(\bar{z}_i^1, \underline{z}_i^1)$ respectively, are perceived by each consumer. Consumers must determine their consumption x_i^1 so as to satisfy the quantity constraints at given prices, say p^1. On the other hand a bond market is perfect; each consumer can choose bond b_i at an announced bond price q^1 without any rationing. That is, consumer i must firstly determine his action under the following budget and quantity constraints; given current signal $s_i^1=(p^1, q^1, \bar{z}_i^1, \underline{z}_i^1)$,

$$(1)\qquad \begin{aligned} &p^1(x_i^1-x_i^0)+q^1b_i-(q^1+1)b_i^0+m_i-m_i^0\leqq 0\\ &(x_i^1, m_i)\in R_+^{l+1},\ b_i\in R \text{ and } \underline{z}_i^1\leqq x_i^1-x_i^0\leqq \bar{z}_i^1.\end{aligned}$$ [2)]

Secondly consumers consider consumption at the next date; consumer i expects the next date's signal $s_i^2=(p^2, q^2, \bar{z}_i^2, \underline{z}_i^2)$ and determines his action under the expected budget and quantity constraints;[3)]

(2) $$p^2x_i^2 \leqq (q^2+1)b_i+m_i$$
$$0 \leqq x_i^2 \leqq \bar{z}_i^2.$$

Thirdly we assume that any consumer must not have defaults in the future. Fomally it is expressed by

(3) $$(q^2+1)b_i+m_i \geqq 0.$$

We suppose that this condition must hold almost everywhere for any given current signals.

Each consumer would expect signals at the next date on the basis of signals at the current date. Let S^1 be the set of signals at the current date in which prices are fixed as $p^1 \gg 0$; *i.e.* $S^1 \equiv \{p^1\} \times R_{++} \times \bar{R}_+^l \times \bar{R}_-^l$.[4] The set of signals at the next date is denoted by $S^2 \equiv R_{++}^{l+1} \times \bar{R}_+^l \times \bar{R}_-^l$. Let $\mu(S^2)$ be the set of probability measures defined on S^2. Consumer i's expectation function ψ_i is represented by $\psi_i: S^1 \rightarrow \mu(S^2)$.

Further his action a_i is defined by $a_i \equiv (x_i^1, b_i, m_i) - e_i$. We assume for each consumer a utility function $u_i: R_+^{2l} \rightarrow R_+$ which is uniformly bounded, strictly monotone, concave and continuous. Let $\phi_i(a_i, s^2)$ be the optimal value of u_i subject to the set $\{(x_i^1, x) \in R_+^{2l} | p^2x \leq (q^2+1)b_i+m_i,\ 0 \leq x \leq \bar{z}_i^2\}$. ϕ_i is well-defined with respect to $s^2 \in S^2$ and $a_i \in R^{l+2}$ and continuous. The expected utility value v_i is therefore defined by

$$v_i(a_i, s_i^1) \equiv \int_{S^2} \phi_i(a_i, .)\, d\psi_i(. ; s_i^1).$$

Thus each consumer determines his action so as to maximize v_i subject to constraints (1) and (3). Let $A_i(s_i^1)$ be the subset of R^{l+2} whose elements satisfy (1) and (3) for all $s^2 \in supp\ \psi_i(s_i^1)$; $supp\ \psi_i(s)$ denotes the support of ψ_i at s. Let $\xi_i(s_i^1)$ be the set of optimal elements in $A_i(s_i^1)$.

(2.1) ψ_i is continuous in the weak topology.

(2.2) $\sigma_i \equiv supp\ \psi_i$ is upper hemi-continuous (u.h.c.) on S^1.

Let $\bar{q}>0$ be such that $S^0 \equiv \{s^1 \in S^1 | q^1 < \bar{q}$ for $s^1 = (p^1, q^1, \bar{z}^1, \underline{z}^1)\}$ and let $B(s^1)\equiv\{s^2\in S^2|\ q^2+1>q^1 \text{ for } s^2=(p^2, q^2, \bar{z}^2, \underline{z}^2)\}$.

(2.3) For each $s\in S^0$, $\psi_i(B(s); s)>0$.

The above condition means that consumer i expects that if any signal with a bond price below $\bar{q}$ is perceived, there will be a positive probability of opportunities such that bond holdings are profitable, *i.e.*, $q^2+1>q^1$.

(2.4) Given the upper bond price $\bar{q}$,

$$(\bar{q}+1)\,b_i^0+m_i^0>0.$$

This condition states that the existing debts are not burdensome so that defaults will be avoided even at higher bond prices. To some extent it corresponds to condition (3) in a stronger sense at the preceding date.

By assuming the above four assumptions consumer i's action set $A_i: S^0\rightarrow R^{l+2}$ satisfies the following property: given $s\in S^0$, $A_i(s)$ is nonempty, convex, compact and continuous. The proofs are almost according to Green [6]. In particular, the proof of the lower hemi-continuity of $A_i(.)$ is shown in Section 6. Since v_i is continuous on $R^{l+2}\times S^1$ (*e.g.* see Grandmont [2]) and concave, it follows:

PROPOSITION 2.1. $\xi_i: S^0\rightarrow R^{l+2}$ is nonempty, convex-, compact-valued and u.h.c..

Next we make a specification on the upper bound of bond price $\bar{q}$ (*i.e.* the lower bound of rate of interest, say $\underline{r}=1/\bar{q}$). As bond price q approaches $\bar{q}$, some consumers expect that the future bond price will fall down almost certainly and furthermore that bond holdings will be unprofitable very probably. For $i\in I$ let $E(s_n^1)\equiv\{s^2\in S^2|\ q^2+1<q_n^1\}$ and $F(\nu)\equiv\{s^2\in S^2|\ \nu\iota\leqq\bar{z}^2\}$, ι=unit vector, provided that $s_n^1=(p^1, q_n^1, \bar{z}_n^1, \underline{z}_n^1)$, $s^2=(p^2, q^2, \bar{z}^2, \underline{z}^2)$ and $a_i^n=(x_i^n, b_i^n, m_i^n)-e_i$. Define $D(s_n^1, \nu)\equiv E(s_n^1)\cap F(\nu)$.

(2.5) For any converging sequence $s_n^1\rightarrow s_0^1$ with $q_n^1\rightarrow\bar{q}$,
(i) $\psi_i(B(s_n^1); s_n^1)\rightarrow 0$, (ii) $\psi_i(D(s_0^1, \nu); s_0^1)>0$ for all $\nu=1, 2, \ldots$, and (iii) $\psi_i(\partial\bar{E}(s_n^1); s_0^1)=0$ for $n=0, 1, 2, \ldots$; $\partial\bar{E}$ denoting the boundary of closure of E.

In particular the condition (iii) implies that the probability with which $q^2+1=\bar{q}$ at the next date will be zero when the current bond price is $\bar{q}$. Under these conditions the demand for money is infinite as $q_n^1 \to \bar{q}$.

PROPOSITION 2.2 Let $\{s_n^1\}$ be a converging sequence with $q_n^1 \to \bar{q}$. Under (2.5) $m_i^n \to +\infty$ for $(x_i^n, b_i^n, m_i^n) - e_i \in \xi_i(s_n^1)$.

(2.6) For any converging sequence $s_n^1 \to s_0^1$ with $q_n^1 \to 0$, there exists a probability measure ϕ_i^0 to which $\phi_i(.\,; s_n^1)$ converges weakly and which satisfies $\phi_i^0(B(s_0^1) \cap F(\nu)) > 0$ for all ν.[5)]

Like (2.5) it also implies that even when bond price q^1 approaches zero, there are probably states not purchase-constrained. In such a case, it is shown, the demand for bond is unbounded.

PROPOSITION 2.3 Let $\{s_n^1\}$ be a converging sequence with $q_n^1 \to 0$. Suppose (2.6) and then $b_i^n \to +\infty$ for $(x_i^n, b_i^n, m_i^n) - e_i \in \xi_i(s_n^1)$ with $m_i^0 > 0$.

III. PRODUCERS

Let J be the set of producers and denote by $Y_j \subset R_+^l \times R_-^l$, $j \in J$, producer j's production set which satisfies the following properties:

(3.1) (i) Y_j is closed and convex; (ii) if y^1 is bounded, any y^2 satisfying $(y^1, y^2) \in Y_j$ is also bounded; (iii) for any $y^1 \in R_+^l$ there exists y^2 such that $(y^1, y^2) \in Y_j$.

Under (3.1) we can define $Y_j(y^1) \equiv \{y^2 \in R_-^l \mid (y^1, y^2) \in Y_j\}$, and let $K_j \subset R_+^l$ be a nonempty compact set. Obviously $Y_j(y^1)$ is nonempty, convex and compact for each $y^1 \in K_j$ by (3.1). The mapping Y_j: $K_j \to R_-^l$ is easily checked to be continuous. We define the producer's profit function by

$$r_j(a_j^1, s^2) \equiv sup\{(q^2+1)b_j - p^2x^2 + m_j \mid x^2 \in Y_j(x_j^1),\ x^2 \geq z^2\}$$

where $a_j^1 = (x_j^1, b_j, m_j) - e_j$ and $s^2 = (p^2, q^2, z^2, z^2)$; $e_j = (x_j^0, b_j^0, m_j^0)$ means producer j's endowment. $r(.)$ is well-defined for each $(a_j^1, s^2) \in R^{l+2}$

$\times S^2$ and also continuous, since $Y_j(y_j^1)$ is compact and continuous by defining K_j well.

Producers act so as to maximize his expected profit under the following constraints:

(4) $$m_j - m_j^0 \leqq (q^1+1)\,b_j^0 - q^1 b_j - p^1(x_j^1 - x_j^0)$$
$$(x_j^1,\ m_j) \in R_+^{l+1},\ b_j \in R \text{ and } \underline{z}_j^1 \leq x_j^1 - x_j^0 \leq \bar{z}_j^1.$$

(5) $$r_j(a_j, s^2) \geqq 0\,.$$

The feasible set A_j is defined by $A_j(s_j^1) \equiv \{a \in R^{l+2} \mid$ (4) and (5) are satisfied for every $s^2 \in supp\ \phi_j(s_j^1)\}$, in which the expectation function $\phi_j : S^1 \rightarrow \mu(S^2)$ is defined similarly to that of consumers. The expected profit v_j is defined by the expectation of producer j's object function $u_j : R \rightarrow R_+$ which is uniformly bounded, monotone, concave and continuous; *i.e.*

$$v_j(a,\ s_j^1) \equiv \int_{S^2} u_j(r_j(a,.))\, d\phi_j(.\ ;\ s_j^1)\,.$$

We denote by $\xi_j(s_j^1)$ the set of optimal elements which maximize $v_j(.,\ s_j^1)$ subject to $A_j(s_j^1)$.

(3.2) (2.1)-(2.4) holds similarly for producer j.

Then the following proposition can be shown in the similar procedure to the case of consumers.

PROPOSITION 3.1. Suppose (3.1) and (3.2) hold for producer j. Then $\xi_j : S^0 \rightarrow R^{l+2}$ is nonempty, convex-, compact-valued and u.h.c..

As mentioned before, producers' objectives are represented by maximization of the expected sum of proceeds, the value of bonds and money at the next date. If money holdings are profitable at the current date (*i.e.* $q^2+1<q^1$) or if bond holdings are profitable ($q^2+1>q^1$), money holdings or bond holdings, respectively, will directly bring about greater (expected) profits to producers. Therefore purchase-constraints at the next date, as denoted by $F(.)$, are unnecessary for them to consider, for they may hold, directly, money and bonds at the next date to gain greater profits. Hence;

(3.3) For any converging sequence $s_n^1 \to s_0^1$ with $q_n^1 \to \bar{q}$,
$\phi_j(B(s_n^1); s_n^1) \to 0$, $\phi_j(E(s_0^1); s_0^1) > 0$ and
$\phi_j(\partial\bar{E}(s_n^1); s_0^1) = 0$ for $n = 0, 1, 2, \ldots$.

Under condition (3.3) the proposition corresponding to Proposition 2.2 holds for producers and, of course, that to Proposition 2.3 holds directly.

IV. BANK

The monetary authority (called *Bank*) has two alternative policies. The one, called an *interest rate rule*, is to determine a rate of interest (or a bond price) as its target and to adjust money supply so as to maintain the rate. The other, called a *money supply rule*, is to set the level of money stock as its target and to determine new money supply so as to attain the level. Under either rule Bank would achieve the target through market operations. It is due to the supposition that a bond market is so perfect at any temporary date as to balance flexibly bond demand and supply by its price.

Let b_B be the quantity of bond desired by Bank and b_B^0 be its initially endowed quantity of bond. In equilibrium it follows:

$$\sum_{i \in I \cup J} (b_i - b_i^0) + (b_B - b_B^0) = 0. \tag{6}$$

It is rewritten as

$$\sum_{i \in I \cup J} b_i + b_B = 0,$$

for this equation holds at the preceding date, *i.e.*, $\sum_i b_i^0 + b_B^0 = 0$. The (new) money supply (or demand) m_B is therefore defined by

$$m_B = -(q^1 b_B - (q^1 + 1) b_B^0). \tag{7}$$

If $m_B < 0$, Bank additionally supplies money and to the contrary, if $m_B > 0$, it withdraws money from circulation. According to Bank's action, the total money stock in an economy M is also defined by

$M = M^0 - m_B$ in which $M^0 \equiv \sum_{i \in I \cup J} m_i^0$. Bank's action a_B is represented by

$$a_B = (0,\ b_B,\ m_B) - (0,\ b_B^0,\ m_B^0). \tag{8}$$

Let $\eta(.,r)$ be the action correspondence of Bank under an interest rate rule, *i.e.*, $\eta(s_B^1, r) \equiv \{a_B \in R^{l+2} \mid a_B$ satisfies (6), (7) and (8) at s_B^1 with fixed $r = 1/q\}$ given $s_B^1 = (p^1, q^1, 0, 0)$. The corresponding money stock $\mu(.)$ is represented by $\mu(s_B^1) \equiv \{M^0 - m_B \mid a_B \in \eta(s_B^1, r)\}$. Obviously $\eta(.,r)$ has the identical properties to $\sum_i \xi_i(.)$. On the other hand let $\eta'(.,M)$ be the action correspondence under a money supply rule, defined by $\eta'(s_B^1, M) \equiv \{a_B \in R^{l+2} \mid a_B$ satisfies (7) and (8), and $m_B = M^0 - M\}$.

V. EQUILIBRIUM

In this section we introduce the notion of equilibrium and show a liquidity trap and the invalidity of monetary policy in addition to the existence of equilibrium. We denote by $a = ((a_i)_{i \in I \cup J}, a_B)$ an *allocation* and by $s = ((s_i)_{i \in I \cup J}, s_B)$ a *signal*, in which $s_i \in S^1$ for every $i \in I \cup J \cup \{B\}$.

DEFINITION. A pair of allocation and signal with a rate of interest (resp. money stock) (a, s, r) (resp. (a, s, M)) is called a fixed price equilibrium under an interest rate rule (resp. a money supply rule) iff

i) $\sum_{i \in I \cup J} a_i + a_B = 0$

ii) $a_i \in \xi_i(s_i)$ for every $i \in I \cup J$ and $a_B \in \eta(s_B, r)$ given a rate of interest r (resp. $a_B \in \eta'(s_B, M)$ given money stock M).

In order to compare the fixed price equilibrium with the Walrasian equilibrium, we define the Walrasian trade correspondence of agent i, say ξ_i^w, by the value of ξ_i in the case of putting $z_i^k = +\infty$ and $z_i^k = -\infty$, $k = 1, 2$. Since no quantity is constrained in the Walrasian actions, $\xi_i(p, q, +\infty, -\infty)$ may be rewritten by $\xi_i^w(p, q)$. Let $a^w = ((a_i^w), a_B)$ be a Walrasian allocation in which $a_i^w \in \xi_i^w(p, q)$, $i \in I \cup J$

and $a_B \in \eta(p, q, 0, 0, r)$ (or $\eta'(p, q, 0, 0, M)$). If $(a^w, (p, q))$ satisfies the conditions of Definition by replacing (ξ_i) by (ξ_i^w), it is called a *Walrasian equilibrium*. In the allocation of a fixed price equilibrium differs from that of a Walrasian equilibrium, the fixed price equilibrium is called *non-Walrasian*. Obviously, if the corresponding prices are not Walrasian equilibrium prices, the equilibrium is trivially non-Walrasian. The following theorems, however, imply that non-Walrasian fixed price equilibria exist at Walrasian equilibrium prices as well as at such non-Walrasian equilibrium ones.

To begin with, we specify the range of quantity constraints with fixed prices.

(5.1) i) $p \gg 0$,

ii) $a_k^+(q) \equiv \max_{i \in I \cup J} (\xi_{ik}^w(p, q), 0) > 0$

$a_k^-(q) \equiv \min_{i \in I \cup J} (\xi_{ik}^w(p, q), 0) < 0$

for each $q \in (0, \bar{q})$ and for $k=1, 2, \ldots l$,

iii) $\#(I \cup J \cup \{B\}) \max_{i \in I \cup J \cup \{B\}} e_i$ is bounded.[6]

THEOREM 5.1 Suppose that (2.1)–(2.4), (3.1) and (5.1) hold for any agent of I and J. Fix $p \gg 0$ and $0 < q < \bar{q}$. Then there exists a fixed price equilibrium under an interest rate rule (a^*, s^*, r^*) which is non-Walrasian and $M < +\infty$ for every $M \in \mu(s_B^*)$. Under the money supply rule the similar proposition holds.

THEOREM 5.2. Suppose that (2.1)–(2.4), (3.1) and (5.1) hold for any agent of I and J and in particular (2.5) holds for some consumers or (3.3) for some producers and (2.6) holds for some agents with $m_i^0 > 0$. Fix $p \gg 0$ and $0 < M < +\infty$. Then there exists a non-Walrasian fixed price equilibrium under a money supply rule (a^{**}, s^{**}, M) such that $r^{**} = 1/q^{**} > \underline{r}$.

Corresponding to these theorems, we show a "liquidity trap" and persistency of non-Walrasian equilibria which are the main subjects. The following proposition implies that the monetary authority cannot move allocations almost anywhere (it fails to move any allocation except in the neighborhood of an allocation).

PROPOSITION 5.3. Suppose the conditions as assumed in Theorem 5.1. Under (2.5) for some consumers or (3.3) for some producers there exists a sequence of non-Walrasian fixed price equilibria under an interest rule $\{(a_n, s_n, r_n)\}$ given $\varepsilon>0$ such that for any n, (i) $x_i^n \in N_\varepsilon(x_i)$ for each $i \in I \cup J$, in which (x_i) is a non-Walrasian allocation, (ii) $M_n \in [M(\varepsilon), +\infty) \subset R_+$ for every $M_n \in \mu(s_B^n)$ and (iii) $M_n \to +\infty$ as $r_n \to \underline{r}$.

The notation is: $a_n = ((a_i^n), a_B^n)$, $a_i^n = (x_i^n, b_i^n, m_i^n) - e_i$ and $s_n = ((s_i^n), s_B^n)$, $s_i^n = (p, q_n, \bar{z}_i^n, \underline{z}_i^n)$ with $r_n = 1/q_n$; $N_\varepsilon(x_i)$ means the ε-neighborhood of x_i.

In the case that Bank acts according to a money supply rule, the similar proposition holds as well.

PROPOSITION 5.4. Suppose the conditions as assumed in Theorem 5.2. Then there exists a sequence of non-Walrasian fixed price equilibria under a money supply rule $\{(a_n, s_n, M_n)\}$ given $\varepsilon>0$ such that for any n, (i) $x_i^n \in N_\varepsilon(x_i)$ for each i and for a non-Walrasian allocation (x_i), and (ii) $r_n \to \underline{r}$ as $M_n \to +\infty$.

The notation is identical to that of Proposition 5.3.

VI. CONCLUDING REMARKS

Let us concentrate upon two points. Firstly we consider whether this model can exhibit a positive aspect of monetary policy as well as the negative one represented by its invalidity under a liquidity trap. For example let us examine the effect of falling a rate of interest. To this, however, we cannot define *a priori* its whole effect (such as one to employment). But motives for further production may be given by the effect on finance at least to producers, as long as the stochastic structure of prices and demands in the future is not so greatly changed by current signals. A falling rate of interest is expected to ease producers' financings by rising bond prices and to bring about greater profits than a remaining rate of interest does.

To increase production will be more profitable as long as sales in the future are probably guaranteed.

By this first effect additional demands for production factors are generated and by interreactions among industries they move an economy toward absorbing excess supplies. But, if producers expect that demands in the future are constrained uniformly at almost every state, the effect on finance by falling a rate of interest is not effective and remains an existing equilibrium. It my recall the interest-inelasticity of investment.

The second remark is concerned with the so-called " transactions motive to liquidity ". In this model no direct transactional constraint is supposed. It implies that, whether or not agents initially hold money, there is no constraint for them to transact at a current date and suggests that there exists a well-organized clearing system which smoothly resolves difficulties on current transactions. But, as Hool [7] exhibited, the conclusion remains, in essence, even assuming transactional constraints in the model such as the Clower ones.

VII. PROOFS

According to Green [6] we can check that for each consumer $A_i(s)$ is nonempty, convex and compact at all $s \in S^1$ if (2.3) and (2.4) hold and that it is upper hemi-continuous if (2.1) and (2.2) hold. We show in the following that $A_i(.)$ is lower hemi-continuous (l.h.c.) on S^1 under (2.1), (2.2) and (2.4).

Proof. In what follows we omit index i. Let $\{s_n\}$ be a converging sequence such that $s_n \to \bar{s}$ and let $\bar{a} \in A_i(\bar{s})$. We wish to show existence of a sequence $\{a_n\}$: $a_n \to \bar{a}$ and $a_n \in A_i(s_n)$, each n. Given b and m, let $f(b, m) \equiv min(q^2+1)b+m$, $q^2 \in Q(s)$ (projection of $supp\ \psi_i(s)$ into R_{++}). Obviously $f(b, m)$ is continuous for (b, m). By (2.4) there exists (b, m) such that $(q^2+1)b+m>0$ for all $q^2 \in Q(\bar{s})$. It implies $f(b, m)>0$. Let $\hat{a}=(x^0, b, m)-e$ and $\tilde{a}=\lambda\bar{a}+(1-\lambda)\hat{a}$, $0<\lambda<1$. If $\tilde{a}=(\tilde{x}, \tilde{b}, \tilde{m})-e$,

$$p(\tilde{x}-x^0)+q^1\tilde{b}-(q^1+1)b^0+\tilde{m}-m^0<0,$$
$$\underline{z}^1\leq\tilde{x}-x^0\leq\bar{z}^1,$$
$$(q^1+1)\tilde{b}+\tilde{m}\geqq(1-\lambda)f(b,m)>0 \quad \text{all } q^2\in Q(\bar{s}).$$

For n large enough

$$p(x_n-x^0)+q_n^1 b_n-(q_n^1+1)b^0+m_n-m^0<0$$

in which x_n satisfies $min\ ||x_n-\tilde{x}||$ subject to $\underline{z}_n^1\leq x_n-x^0\leq\bar{z}_n^1$ and $a_n=(x_n, b_n, m_n)-e$ is at the neighborhood of a. Since $supp\ \phi_i$ is continuous, $(q^2+1)b_n+m_n>0$ at all $q^2\in Q(s_n)$ for n large enough. It implies that there exists a sequence $\{a_n\}$ such that $a_n\rightarrow\tilde{a}$, $a_n\in A_i(s_n)$ and $\tilde{a}\in A_i(\bar{s})$, as $s_n\rightarrow\bar{s}$. By taking adequately $\{\lambda_n\}$ such as $\lambda_n\rightarrow 1$, it follows $a_n\rightarrow\bar{a}$. Q.E.D.

Proof of Proposition 2.2. To the contrary suppose that m_i^n is bounded for every n and therefore b_i^n bounded. Let $b_i=\lim_n b_i^n$, $\tilde{b}_i^n=b_i^n-\delta$, $\delta>0$, and $\tilde{m}_i^n=m_i^n+q_n\delta$. Let $\tilde{a}_i^n=(x_i^n, \tilde{b}_i^n, \tilde{m}_i^n)-e_i$: it is constructed so as to satisfy the budget condition (1).

$$G_i^n\equiv\int_{S^2}u_i^n\, d\phi_i^n-\int_{S^2}\tilde{u}_i^n\, d\phi_i^n,$$

in which $\phi_i^n\equiv\phi_i(.\,; s_i^n)$, $u_i^n\equiv\phi_i(a_i^n,.)$ and $\tilde{u}_i^n\equiv\phi_i(\tilde{a}_i^n,.)$, $\Delta u_i^n(.)\equiv u_i^n(.)-\tilde{u}_i^n(.)$. Remark $u_i^n(s^2)\geqq u_i(x_i^n, 0)\equiv u^n(0)$ for all $s^2\in S^2$. Below we omit subscript i.

$$G_i^n=\int_{\bar{E}^n}\Delta u^n d\phi^n+\int_{B^n}\Delta u^n d\phi^n+\int_{S^2\setminus\bar{E}^n\cup B^n}\Delta u^n d\phi^n \tag{7.1}$$
$$\leqq\int_{\bar{E}^n}\Delta u^n d\phi^n+\int_{B^n}(u^n-u^n(0))\, d\phi^n,$$ [7]

in which $\bar{E}^n\equiv\bar{E}(s_i^n)$, $B^n\equiv(s_i^n)$ and $D^n(\nu)\equiv D(s_i^n, \nu)$. The second inequality is derived from

$$(q^2+1)\tilde{b}^n+\tilde{m}^n=(q^2+1)b^n+m^n+[q_n-(q^2+1)]\delta.$$

Similarly it follows: there is $\bar{\nu}_n$ such that

$$\tilde{u}^n(s^2)>u^n(s^2) \quad \text{at all } s^2\in D^n(\nu) \quad \text{and}$$
$$\text{for all } \nu\geqq\bar{\nu}_n.$$

By assumption $\tilde{a}^n$ and a^n are bounded for all n. If $A^2(a, s^2) \equiv \{x \in R^l \mid p^2 x \leqq (q^2+1) b + m,\ 0 \leq x \leq \bar{z}^2\}$, then it is checked by (2.5) that $\lim_n A^2(a^n, s^2) \subset \lim_n A^2(\tilde{a}^n, s^2)$, *i.e.*, there exist some $x \in \lim_n A^2(a^n, s^2)$ and some $\tilde{x} \in \lim_n A^2(\tilde{a}^n, s^2)$ at all $s^2 \in D(\nu\,;\, s_i^0)$, $s_i^n \to s_i^0$, for $\nu \geqq \bar{\nu}$ such that $\bar{\nu}_\iota > \tilde{x} \gg x$. Therefore, $\lim_n u^n(s^2) \leqq \lim_n \tilde{u}^n(s^2)$ at all $s^2 \in \lim_n \bar{E}^n$, and $\lim_n u^n(s^2) < \lim_n \tilde{u}^n(s^2)$ at all $s^2 \in D(\nu\,;\, s_i^0)$, all $\nu \geqq \bar{\nu}$ by the monotonicity. By (7.1) it follows:

$$G_i^n \leqq \int_{\bar{E}^n} \Delta u^n d\phi^n + K\phi(B^n\,;\, s_i^n),$$

since $\sup_{s \in S^2} (u^n(s) - u^n(0)) \leqq K$ for some $K > 0$ by the uniformly bounded utility function.

Let $\bar{E} \equiv \lim_n \bar{E}^n$, $\Delta u \equiv \lim_n \Delta u^n$ and $\phi \equiv \phi(\cdot\,;\, s_i^0)$.[8)]

$$
\begin{aligned}
&\left| \int_{\bar{E}^n} \Delta u^n d\phi^n - \int_{\bar{E}} \Delta u d\phi \right| \\
&\leqq \left| \int_{\bar{E}^n} \Delta u^n d\phi^n - \int_{\bar{E}} \Delta u^n d\phi^n \right| + \left| \int_{\bar{E}} \Delta u^n d\phi^n - \int_{\bar{E}} \Delta u d\phi \right| \\
&\leqq K(\phi^n(\bar{E}^n \cup \bar{E}) - \phi^n(\bar{E}^n \cap \bar{E})) + \left| \int_{\bar{E}} \Delta u^n d\phi^n - \int_{\bar{E}} \Delta u d\phi \right|.
\end{aligned}
$$

Since $\bar{q} > q_n^1$ all n and $q_n^1 \to \bar{q}$, there exists a subsequence $\{q_m^1\}$ such that $q_m^1 \leqq q_{m+1}^1$ and therefore $\bar{E}^m \subset \bar{E}^{m+1}$ for all m. Remark $\lim_m (\bar{E}^m \cup \bar{E}) = \lim_m (\bar{E}^m \cap \bar{E}) = \bar{E} = \bar{E}(s_1^0)$, $\bar{E}^m \subset \bar{E}$, all m.

The limit of the first term $\leqq \lim_m (\lim_n K(\phi^n(\bar{E}^m \cup \bar{E}) - \phi^n(\bar{E}^m \cap \bar{E}))$

$$
\begin{aligned}
&= K \lim_m (\lim_n (\phi^n(\bar{E}) - \phi^n(\bar{E}^m))) \\
&= K \lim_m (\phi(\bar{E}\,;\, s_1^0) - \phi(\bar{E}^m\,;\, s_1^0)) \\
&= 0 \quad \text{for } s_1^0 = \lim_n s_1^n.
\end{aligned}
$$

The third equation holds on $\bar{E}$ and $\bar{E}^m$ by (2.5) (iii) and convergence criteria [9, 12–1, p. 190] and the fourth does by $\lim_m \bar{E}^m = \bar{E}$. Further we know that $\Delta u^n \to \Delta u$ pointwise and Δu^n (Δu) is uniformly bounded. Hence by Delbaen [1] the second term also converges to zero. Since $\phi(D(\nu, s_1^0)\,;\, s_1^0) > 0$ for all ν from (2.5) (ii) and $D(\nu\,;\, s_1^0) \subset \bar{E} = \bar{E}(s_1^0)$, all ν, then $\int_{\bar{E}} \Delta u d\phi < 0$. Also $K\phi(B^n\,;\, s_i^n) \to 0$ by (2.5) (i). Thus there exists $n_0>0$ such that for every $n \geqq n_0$ $G_i^n<0$. It contradicts $a_i^n \in \xi_i(s_i^n)$, all n. Q.E.D.

Proof of Proposition 2.3. For n large enough, consider an action a_i^n which is similar to that of Proposition 2.2 except setting $\delta<0$. Remark that $\psi_i(E(s_n)\,;\,s_n)=0$ for n large enough. By assumption $\psi_i^0(B(s_0)\cap F(\nu))>0$, all ν, for $s_n \to s_0$. Therefore we can prove the proposition in the similar procedure to Proposition 2.2. Q.E.D.

Proof of Theorem 5.1. The essence of the proof is due to Grandmont and Laroque [5]. Given $\varepsilon>0$ we define

$$P_\varepsilon \equiv \{\pi \in R_+^l \mid |\pi_k - p_k| \leqq \varepsilon, \text{ all } k\}.$$

Let w_i^+ and w_i^- be, respectively, such that $w_{ik}^+ = min(h_k, z_{ik}^{w+}(p, q))$ and $w_{ik}^- = \max(-h'_k, z_{ik}^{w-}(p, q))$ for each k, in which $z_{ik}^{w+}(p, q) = \max\{0, \xi_{ik}^w(p, q)\}$ and $z_{ik}^{w-}(p, q) = \min\{0, \xi_{ik}^w(p, q)\}$ and in which we set $h=(h_k)$, $h'=(h'_k) \gg 0$ such that $h_k < z_{ik}^{w+}(p, q)$ for some i and k and $-h'_k > z_{ik}^{w-}(p, q)$ for some i and k. It is possible by assumption (5.1). Let $\bar{z}_i : P_\varepsilon \to R_+^l$ and $\underline{z}_i : P_\varepsilon \to R_-^l$ be bounded continuous functions such that $\bar{z}_{ik}(\pi) = w_{ik}^+$ if $\pi_k \leqq p_k$, $\bar{z}_{ik}(\pi)=0$ if $\pi_k = p_k + \varepsilon$ and $\underline{z}_{ik}(\pi) = w_{ik}^-$ if $\pi_k \geqq p_k$, $\underline{z}_{ik}(\pi)=0$ if $\pi_k = p_k - \varepsilon$. Let $s_i(\pi) \equiv (p, q, \bar{z}_i(\pi), \underline{z}_i(\pi))$, and $\xi(\pi) \equiv \sum_{i \in I \cup J} \xi_i(s_i(\pi)) + \eta_B$.

$\zeta(P_\varepsilon)$ can be included by a compact, convex set of R^{l+2}. With Propositions 2.1 and 3.1, we can apply the proof of Grandmont and Laroque [5, p. 65] to this case. Hence we have $0 \in \zeta(\pi^*)$, $\pi^* \in P_\varepsilon$. Since $z_{ik}(\pi^*) < z_{ik}^{w+}(p, q)$ or $z_{ik}(\pi^*) > z_{ik}^{w-}(p, q)$ for some i and for some k, the corresponding allocation is non-Walrasian.

Suppose that M is unbounded for $M \in \mu(s^*)$, $s^* = ((s_i^*)_{i \in I \in J}, s_B^*)$. $s_i^* = (p, q, \bar{z}_i(\pi^*), \underline{z}_i(\pi^*))$. The m_i^* is unbounded for some i where $a_i^* = (x_i^* - x_i^0, b_i^* - b_i^0, m_i^* - m_i^0) \in \xi_i(s_i^*)$. Since $m_i^* - m_i^0 \leqq -p(x_i^* - x_i^0) - q^1 b_i^* + (q^1+1) b_i^0$ and since $x_i^* - x_i^0$ is bounded and $q^1>0$, $b_i^* \to -\infty$ as $m_i^* \to \infty$.

$$\begin{aligned}(q^2+1)b_i^* + m_i^* &\leqq (q^2+1)b_i^* + m_i^0 - p(x_i^* - x_i^0) - q^1 b_i^* + (q^1+1)b_i^0 \\ &= (q^2+1-q^1)b_i^* + d_i\end{aligned}$$

in which d_i is finite. As long as $q^1<\bar{q}$, $\psi_i(B(s_i^*); s_i^*)>0$ by (2.3). Then $(q^2+1)b_i^*+m_i^*\to-\infty$ for some $s^2\in supp\ \psi_i(s_i^*)$, as $m_i^*\to\infty$. It contradicts (3). Therefore M is bounded. Q.E.D.

Proof of Theorem 5.2. Suppose $s_i'(\pi')=(p, q, \bar{z}_i(\pi), \underline{z}_i(\pi))$, $\pi'=(\pi, q)$ in which $(\bar{z}_i(.), \underline{z}_i(.))$ is defined similarly to Theorem 5.1. Let us define $\zeta'(\pi')\equiv\sum_{i\in I\cup J}\xi_i(s_i'(\pi'))+\eta'(s_B^1, M)$. If $P_\varepsilon^n\equiv\{(\pi, q)\mid\pi\in P_\varepsilon, q\in L^n\equiv[1/Ln, \bar{q}-1/Ln]\}$, where $q>1/2L$, $L>0$, then $\zeta'(P_\varepsilon^n)$ is included by a compact, convex set for each n. Let A^n be the set for n. Let $\phi^n(a)\equiv\{(\pi, q)\in P_\varepsilon^n\mid\pi z_x\geqq\rho z_x$ for any $\rho\in P_\varepsilon$, $qz_b\geqq q'z_b$ for any $q'\in L^n\}$ for $a=(z_x, z_b, z_m)\in A^n$. Obviously the product mapping $\zeta'X\phi^n: P_\varepsilon^n\times A^n\to P_\varepsilon^n\times A^n$ satisfies the conditions of Kakutani's theorem; *i.e.*, $(a_n, \pi_n')\in\zeta'(\pi_n')\times\phi^n(a_n)$. Similarly to Theorem 5.1, it follows: $z_x^n=0$, all n for $a_n=(z_x^n, z_b^n, z_m^n)$; if $z_b^n>0$, then $q_n=\bar{q}-1/Ln$ and if $z_b^n<0$, then $q_n=1/Ln$ for $\pi_n'=(\pi_n, q_n)$. Since (π_n') is bounded, we may suppose $\pi_n'\to\pi_0'$ and $a_n\to a_0=(z_x^0, z_b^0, z_m^0)$. We know $z_x^0=0$. Suppose $z_b^0>0$, and then $q_0=\lim_n q_n=\bar{q}$ and $z_m^0<0$ by Walras' law. But it contradicts Proposition 2.2. Conversely suppose $z_b^0<0$, and then $q_0=0$. But $\lim_n z_b^n=+\infty$ by Proposition 2.3. It is a contradiction. Therefore $z_b^0=0$. By Walras' law $z_m^0=0$. Hence $a_0=0$ and $0<q_0<\bar{q}$. By the upper hemi-continuity of ζ', $0\in\zeta'(\pi_0')$, $\pi_0'=\lim_n\pi_n'$. Q.E.D.

Proof of Proposition 5.3. Let $\{q_n\}$ be a converging sequence such that $q_n<\bar{q}$ for all n and $q_n\to\bar{q}$. By using Theorem 5.1 we can yield a sequence of equilibria $\{(a_n, s_n, r_n)\}$ given a sequence of fixed prices and bond prices $\{(p, q_n)\}$, $p\gg0$. If $a_n=(a_i^n)$, $a_i^n=(x_i^n, b_i^n, m_i^n)$ $-e_i$ and if $A_i\equiv\{x\in R^l\mid-h'\leq x-e_i\leq h\}$ as assumed in the proof of Theorem 5.1, then $(x_i^n)\in A\equiv\times_{i\in I\cup J}A_i$, all n. Since A is a nonempty, compact set, there exists a converging subsequence $(x_i^k)\to(x_i)$, $(x_i)\in$ A. Given $\varepsilon>0$ let n_0 be such that $|x_i^k-x_i|<\varepsilon$, all i, for any $k\geqq n_0$. A (sub)sequence of equilibria $\{(a_k, s_k, r_k)|k\geqq n_0\}$ satisfies the conditions of (i) of Proposition 5.3 with help of Theorem 5.1.

By Theorem 5.1, $M^k<+\infty$ for $M^k\in\mu(s_B^k)$ if $r^k>\underline{r}$, in which

$s_k=((s_i^k), s_B^k)$, $r^k=1/q^k$ and $\underline{r}=1/\bar{q}$. Let $M(\varepsilon)\equiv\inf_{k\geq n_0}\mu(s_B^k)<\infty$.

To the contrary suppose $\lim_n M^n<+\infty$, $M^n\in\mu(s_B^n)$. It implies $\lim_n m_B^n$ and so $\lim_n q_n b_B^n$ are bounded, in which $a_B^n=(0, b_B^n, m_B^n)$. Since $q_n>0$, all n, it implies $\lim_n b_i^n$ is bounded for all i and therefore $\lim_n m_i^n$ bounded. But it contradicts Proposition 2.2. Q.E.D.

Proof of Proposition 5.4. Let $\{M_n\}$ be a sequence such that $M_n\rightarrow+\infty$. Correspondingly, by Theorem 5.2, there exists a sequence of fixed price equilibria $\{(a_n, s_n, M_n)\}$, in which $a_B^n\in\eta'(s_B^n, M_n)$, all n. The first proposition is proved similarly to the first proof of Proposition 5.3.

Since q_n is bounded, all n, we may suppose q_n converges to $q^{**}\in[0, \bar{q}]$. Suppose $q^{**}\neq\bar{q}$, and then by Theorem 5.1 $\lim_n M_n$ is bounded. But it is a contradiction. Therefore $q_n\rightarrow\bar{q}$, *i.e.*, $r_n\rightarrow\underline{r}$.

Q.E.D.

FOOTNOTES

1) This simplification may not give an answer to the problem generated by debts *i.e.* bankruptcy. However, bankruptcy is a greatly difficult, unsolved problem to general equilibrium theory, enough to form an independent paper. This paper proceeds by remarking that the primary subject is represented by liquidity trap and monetary policy.
2) Given $x, y\in R^l$, $x=(x_i)$ and $y=(y_i)$, $x\gg y$ means $x_i>y_i$, all i; $x>y$ means $x_i\geqq y_i$, all i and $x_i>y_i$ some i; $x\geq y$ means $x_i\geqq y_i$, all i.
3) One must not consider that any agents' time-perspectives are limited by two periods. Technically it is also possible that the perspectives are extended to multi-periods. Two-periods-horizons are required only for simplicity. To this case it is more appropriate to consider that the 'next date' represents the 'future' for any agents' consumption plans.
4) $\bar{R}^l_{+(-)}$ denotes the positive (negative) orthant of the extended real space $\bar{R}^l$ whose coordinates are composed by an extended real line $\bar{R}=R\cup\{\infty, -\infty\}$. Any $x\in R^l_{++}$ implies $x\gg 0$.
5) Such a weakly converging sequence exists if the family $\{\psi_i(.\,;s)\ s\in S^0\}$ is tight.
6) ξ_{ik}^w denotes the k-th coordinate elements of ξ_i^w. $\# I$ denotes the cardinal number of set I. Remark also that the assumption of finite agents can be relaxed into that of infinite agents by taking the mean values. ($e_B=(0, b_B^0, m_B^0)$.)

7) $\bar{E}^{(n)}$ denotes the closure of $E^{(n)}$ with respect to the topology on S^2.

8) $\underset{n}{Lim}\ \bar{E}^n = \underset{n}{limsup}\ \bar{E}^n = \underset{n}{liminf}\ \bar{E}^n$. Since $A^2(a, s)$ and $\bar{E}(s)$ are continuous (and compact), such convergences can hold.

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